

# Inhomogeneous Long-Range Percolation for Real-Life Network Modeling

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## Abstract

The study of random graphs has become very popular for real-life network modeling such as social networks or financial networks. Inhomogeneous long-range percolation (or scale-free percolation) on the lattice  $\mathbb{Z}^d$ ,  $d \geq 1$ , is a particular attractive example of a random graph model because it fulfills several stylized facts of real-life networks. For this model various geometric properties such as the percolation behavior, the degree distribution and graph distances have been analyzed. In the present paper we complement the picture about graph distances. Moreover, we prove continuity of the percolation probability in the phase transition point.

*Keywords:* Network modeling; stylized facts of real-life networks; small-world effect; long-range percolation; scale-free percolation; graph distance; phase transition; continuity of percolation probability; inhomogeneous long-range percolation; infinite connected component

## 1 Introduction

Random graph theory has become very popular to model real-life networks. Real-life networks may be understood as sets of particles that are possibly linked with each other. Such networks appear for example as virtual social networks, see [19], or financial networks such as the banking system where banks exchange lines of credits with each other, see [3] and [11]. Many different random graph models have been developed in recent years in order to understand the geometry of such networks. Using empirical data one has observed several stylized facts about large real-life networks, for a detailed outline we refer to [19] and Section 1.3 in [15]:

- Distant particles are typically connected by very few links. This is called the “small-world effect”. For example, there is the observation that most particles in real-life networks are connected by at most six links, see also [23].

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- Linked particles tend to have common friends. This is called the “clustering property”.
- The degree distribution, that is, the distribution of the number of links of a given particle, is heavy-tailed, i.e. its survival probability has a power law decay. It is observed that in real-life networks the (power law) tail parameter  $\tau$  is often between 1 and 2, for instance, for the movie actor network  $\tau$  is estimated to be around 1.3. For more explicit examples we refer to [15].

A well studied model in the literature is the homogeneous long-range percolation model on  $\mathbb{Z}^d$ ,  $d \geq 1$ . In this model, the particles are the vertices of  $\mathbb{Z}^d$ . Any two particles  $x, y \in \mathbb{Z}^d$  are linked with probability  $p_{xy}$  which behaves as  $\lambda|x - y|^{-\alpha}$  for  $|x - y| \rightarrow \infty$ . This model has by definition of  $p_{xy}$  a local clustering property. Moreover, if  $\alpha \in (d, 2d)$  the graph distance between  $x, y \in \mathbb{Z}^d$ , that is, the minimal number of links that connect  $x$  and  $y$ , behaves roughly as  $(\log|x - y|)^{1/\log_2(2d/\alpha)}$  as  $|x - y| \rightarrow \infty$ , see [8]. This behavior can be interpreted as a version of the small-world effect. But homogeneous long-range percolation does not fulfill the stylized fact of having a heavy-tailed degree distribution. Therefore, [13] introduced the inhomogeneous long-range percolation model (also known as scale-free percolation model) on  $\mathbb{Z}^d$  which extends the homogeneous model in such a way that the degree distribution turns out to be heavy-tailed. In the inhomogeneous long-range percolation model one assigns to each particle  $x \in \mathbb{Z}^d$  a positive random weight  $W_x$  whose distribution is heavy-tailed with tail parameter  $\beta > 0$ . These weights make particles more or less attractive, i.e. if a given particle  $x$  has a large weight  $W_x$  then it plays the role of a hub in the network. Given these weights, two particles  $x, y \in \mathbb{Z}^d$  are then linked with probability  $p_{xy}$  which is approximately  $\lambda W_x W_y |x - y|^{-\alpha}$  for large  $|x - y|$ . For  $\min\{\alpha, \beta\alpha\} > d$  the degree distribution is heavy-tailed with tail parameter  $\tau = \beta\alpha/d > 1$ , see Theorem 2.2 in [13]. Hence, this model fulfills the stylized fact of having a heavy-tailed degree distribution. For real-life applications the interesting case is  $\tau = \beta\alpha/d \in (1, 2)$ , and in this case, if in addition  $\alpha > d$ , the graph distance between  $x, y \in \mathbb{Z}^d$  is of order  $\log \log |x - y|$  as  $|x - y| \rightarrow \infty$ , see [13] and Theorem 8 below. This is again a version of the small-world effect. One goal of this paper is to complement the picture about graph distances of [13] by providing analogous results to [6, 8, 9, 21] for inhomogeneous long-range percolation.

In homogeneous long-range percolation it is known that there is a critical constant  $\lambda_c = \lambda_c(\alpha, d)$  such that there is an infinite connected component of particles for  $\lambda > \lambda_c$  and there is no such component for  $\lambda < \lambda_c$ , i.e. in the former case there is an infinite connected network in  $\mathbb{Z}^d$ . This phase transition picture in homogeneous long-range percolation can be traced back to the work of [2, 18, 20]. Later work concentrated more on the geometrical properties of percolation like graph distances, see [4, 8, 9, 12, 21]. A good overview of the literature for long-range percolation is provided in [8, 10]. For homogeneous long-range percolation it is known that for  $\alpha \leq d$  there is an infinite connected component for all  $\lambda > 0$ , and therefore  $\lambda_c = 0$ . This infinite connected component contains all particles of  $\mathbb{Z}^d$ , a.s., i.e. in that case we have a completely connected network of all particles of  $\mathbb{Z}^d$ . The case  $\alpha \in (d, 2d)$  is treated in [5]. In that case there is no

infinite connected component at criticality  $\lambda_c$ . This result combined with Proposition 1.3 of [1] shows continuity of the percolation probability, that is, the probability that a given particle belongs to an infinite connected component. For  $\alpha \geq 2d$ , the problem is still open, except in the case  $d = 1$  and  $\alpha > 2$  because in that latter case there does not exist an infinite connected component for any  $\lambda > 0$ .

In inhomogeneous long-range percolation the conditions for the existence of a non-trivial critical value  $\lambda_c \in (0, \infty)$  were derived in [13], see also Theorems 1 and 2 below. The continuity of the percolation probability was conjectured in that article. One main goal of the present work is to prove this conjecture for  $\alpha \in (d, 2d)$ . The crucial technique to prove this conjecture is the renormalization method presented in [5]. This technique will also allow to complement the picture of graph distances provided in [13].

**Organization of this article.** In Section 2, we describe the model assumptions and notations. We also state the conditions that are required for a non-trivial phase transition. In Section 3, we state the main results of the article. Namely, we show the continuity of the percolation function in Theorem 5 which is based on a finite box estimate stated in Theorem 3. We also complement the picture about graph distances of [13], see Theorem 8 below. In Section 4, we discuss open problems and compare the results to homogeneous long-range percolation model results. Finally, we provide all proofs of our results in Section 5.

## 2 Model assumptions and phase transition picture

We define the inhomogeneous long-range percolation model of [13] in a slightly modified version. The reason for this modification is that the model becomes easier to handle but it keeps the essential features of inhomogeneous long-range percolation. In particular, all results of [13] only depend on the asymptotic behavior of survival probabilities. Therefore, we choose an explicit example which on the one hand has the right asymptotic behavior and on the other hand is easy to handle. This, of course, does not harm the generality of the results.

Consider the lattice  $\mathbb{Z}^d$  for fixed  $d \geq 1$  with vertices  $x \in \mathbb{Z}^d$  and edges  $(x, y)$  for  $x, y \in \mathbb{Z}^d$ . Assume  $(W_x)_{x \in \mathbb{Z}^d}$  are i.i.d. Pareto distributed weights with parameters  $\theta = 1$  and  $\beta > 0$ , i.e., the weights  $W_x$  have i.i.d. survival probabilities

$$\mathbb{P}[W_x > w] = w^{-\beta}, \quad \text{for } w \geq 1.$$

Conditionally given these weights  $(W_x)_{x \in \mathbb{Z}^d}$ , we assume that edges  $(x, y)$  are independently from each other either occupied or vacant. The conditional probability of an occupied edge between  $x$  and  $y$  is chosen as

$$p_{xy} = 1 - \exp\left(-\frac{\lambda W_x W_y}{|x - y|^\alpha}\right), \quad \text{for fixed given parameters } \alpha, \lambda \in (0, \infty). \quad (1)$$

For  $|\cdot|$  we choose the Euclidean norm. If there is an occupied edge between  $x$  and  $y$  we write  $x \Leftrightarrow y$ ; if there is a finite connected path of occupied edges between  $x$  and  $y$  we write  $x \leftrightarrow y$

and we say that  $x$  and  $y$  are connected. Clearly  $\{x \Leftrightarrow y\} \subset \{x \leftrightarrow y\}$ . We define the cluster of  $x \in \mathbb{Z}^d$  to be the connected component

$$\mathcal{C}(x) = \{y \in \mathbb{Z}^d; x \leftrightarrow y\}.$$

Our aim is to study the size of the cluster  $\mathcal{C}(x)$  and to investigate its percolation properties as a function of  $\lambda > 0$  and  $\alpha > 0$ , that is, as a function of the edge probabilities  $(\lambda, \alpha) \mapsto p_{xy} = p_{xy}(\lambda, \alpha)$ . Note that  $\mathcal{C}(x)$  exactly denotes all particles  $y \in \mathbb{Z}^d$  which can be reached within the network whose links are described by the occupied edges. The percolation probability is defined by

$$\theta(\lambda, \alpha) = \mathbb{P}[|\mathcal{C}(0)| = \infty].$$

This is non-decreasing in  $\lambda$  and non-increasing in  $\alpha$ . For given  $\alpha > 0$ , the critical value  $\lambda_c(\alpha)$  is defined as

$$\lambda_c = \lambda_c(\alpha) = \inf \{\lambda > 0; \theta(\lambda, \alpha) > 0\}.$$

Note that  $\theta(\lambda, \alpha)$  and  $\lambda_c(\alpha)$  also depend on  $\beta$ , but this parameter will be kept fixed.

*Trivial case.* For  $\min\{\alpha, \beta\alpha\} \leq d$ , we have  $\lambda_c = 0$ . This comes from the fact that for any  $\lambda > 0$

$$\mathbb{P}\left[|\{y \in \mathbb{Z}^d; 0 \leftrightarrow y\}| = \infty\right] = 1,$$

see Theorem 2.1 in [13]. This says that the degree distribution of a given vertex is infinite, a.s., and therefore there is an infinite connected component, a.s. For this reason we only consider the non-trivial case  $\min\{\alpha, \beta\alpha\} > d$  (where a phase transition may occur). In this latter case the degree distribution is heavy-tailed with tail parameter  $\tau = \beta\alpha/d > 1$ , see Theorem 2.2 of [13].

**Theorem 1 (upper bounds)** *Fix  $d \geq 1$ . Assume  $\min\{\alpha, \beta\alpha\} > d$ .*

- (a) *If  $d \geq 2$ , then  $\lambda_c < \infty$ .*
- (b) *If  $d = 1$  and  $\alpha \in (1, 2]$ , then  $\lambda_c < \infty$ .*
- (c) *If  $d = 1$  and  $\min\{\alpha, \beta\alpha\} > 2$ , then  $\lambda_c = \infty$ .*

Since  $W_x \geq 1$ , a.s., the edge probability stochastically dominates a configuration with independent edges being occupied with probabilities  $1 - \exp(-\lambda|x-y|^{-\alpha})$ . The latter is the homogeneous long-range percolation model on  $\mathbb{Z}^d$  and it is well known that this model percolates (for  $d \geq 2$  see [5]; for  $d = 1$  and  $\alpha \in (1, 2]$  see [18]). For part (c) of the theorem we refer to Theorem 3.1 of [13]. The next theorem follows from Theorems 4.2 and 4.4 of [13].

**Theorem 2 (lower bounds)** *Fix  $d \geq 1$ . Assume  $\min\{\alpha, \beta\alpha\} > d$ .*

- (a) *If  $\beta\alpha < 2d$ , then  $\lambda_c = 0$ .*
- (b) *If  $\beta\alpha > 2d$ , then  $\lambda_c > 0$ .*

Theorems 1 and 2 give the phase transition pictures for  $d \geq 1$ , see Figure 1 for an illustration. They differ for  $d = 1$  and  $d \geq 2$  in that the former has a region where  $\lambda_c = \infty$  and the latter does not. Note that  $\beta\alpha < 2d$  corresponds to infinite variance of the degree distribution and  $\beta\alpha > 2d$  to finite variance of the degree distribution. In particular, for the interesting case  $\tau = \beta\alpha/d \in (1, 2)$  we have  $\lambda_c = 0$ , which implies that for any  $\lambda > 0$  the network will have an infinite connected component, a.s.

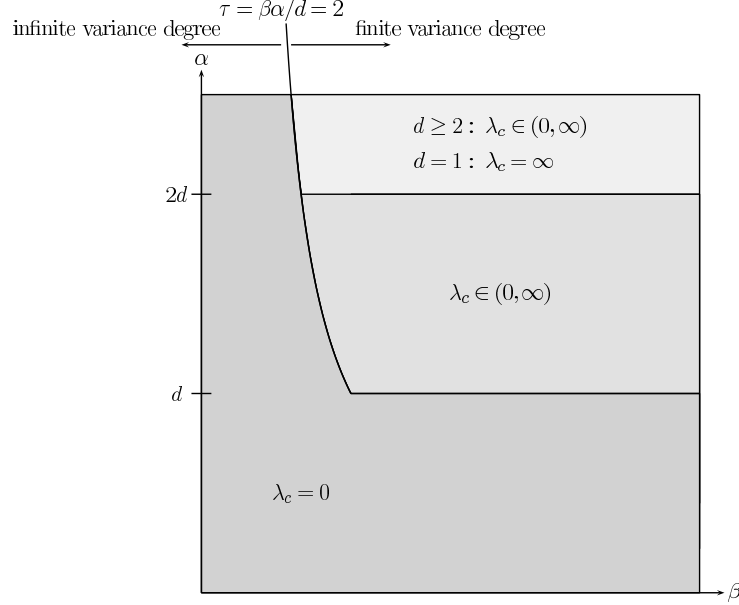


Figure 1: phase transition picture for  $d \geq 1$ .

### 3 Main results

#### 3.1 Continuity of percolation probability

We say that there exists an infinite cluster  $\mathcal{C}$  if there is an infinite connected component  $\mathcal{C}(x)$  for some  $x \in \mathbb{Z}^d$ . Since the model is translation invariant and ergodic, the event of having an infinite cluster  $\mathcal{C}$  is a zero-one event. Thus, for  $\lambda > \lambda_c$  there exists an infinite cluster, a.s. Moreover, from Theorem 1.3 in [5] we know that an infinite cluster is unique, a.s. This justifies the notation  $\mathcal{C}$  for *the* infinite cluster in the case of percolation  $\theta(\lambda, \alpha) > 0$  and implies that we have a unique infinite connected network, a.s.

**Theorem 3** *Assume  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$ . Choose  $\lambda \in (0, \infty)$  with  $\theta(\lambda, \alpha) > 0$ . There exist  $\lambda' \in (0, \lambda)$  and  $\alpha' \in (\alpha, 2d)$  such that*

$$\theta(\lambda', \alpha') > 0.$$

*In particular,  $\{\lambda \in (0, \infty); \theta(\lambda, \alpha) > 0\}$  is an open interval in  $(0, \infty)$ , and there does not exist an infinite cluster  $\mathcal{C}$  at criticality  $\lambda_c$ .*

Note that for  $\beta\alpha < 2d$  we have  $\lambda_c = 0$ , therefore Theorems 2 and 3 imply the following corollary.

**Corollary 4** *Assume  $\alpha \in (d, 2d)$  and  $\tau = \beta\alpha/d > 2$ . There is no infinite cluster  $\mathcal{C}$  at criticality  $\lambda_c > 0$ .*

Next we state continuity of the percolation probability in  $\lambda$  which was conjectured in [13].

**Theorem 5** *For  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$ , the percolation probability  $\lambda \mapsto \theta(\lambda, \alpha)$  is continuous.*

### 3.2 Percolation on finite boxes

For integers  $m \geq 1$  we define the box of size  $m^d$  by  $B_m = [0, m-1]^d$ , and by  $C_m$  we denoted the largest connected component in box  $B_m$  (with a fixed deterministic rule if there is more than one largest connected component in  $B_m$ ).

**Theorem 6** *Assume  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$ . Choose  $\lambda \in (0, \infty)$  with  $\theta(\lambda, \alpha) > 0$ . For each  $\alpha' \in (\alpha, 2d)$  there exist  $\rho > 0$  and  $N_0 < \infty$  such that for all  $m \geq N_0$  we have*

$$\mathbb{P}[|C_m| \geq \rho|B_m|] \geq 1 - e^{-\rho m^{2d-\alpha'}}.$$

This statement says that in case of percolation largest connected components in finite boxes cover a positive fraction of these box sizes with high probability for large  $m$ , or in other words, the number of particles belonging to the largest connected network in  $B_m$  is proportional to  $m^d$ . This is the analog to the statement in homogeneous long-range percolation, see Theorem 3.2 in [8].

For integers  $n \geq 1$  and  $x \in \mathbb{Z}^d$  define the box centered at  $x$  with total side length  $2n$  by  $\Lambda_n(x) = x + [-n, n]^d$  and let  $\mathcal{C}_n(x)$  be the vertices in  $\Lambda_n(x)$  that are connected with  $x$  within box  $\Lambda_n(x)$ . For  $\ell < n$  and  $\rho > 0$  we denote by

$$\mathcal{D}_n^{(\rho, \ell)} = \{x \in \Lambda_n(0); |\mathcal{C}_\ell(x)| \geq \rho|\Lambda_\ell(x)|\}$$

the set of vertices  $x \in \Lambda_n(0)$  which are  $(\rho, \ell)$ -dense, i.e., surrounded by sufficiently many connected vertices in  $\Lambda_\ell(x)$ , see also Definition 2 in [8].

**Corollary 7** *Under the assumptions of Theorem 6 we have the following.*

(i) *There exists  $\rho > 0$  such that for any  $x \in \mathbb{Z}^d$*

$$\lim_{n \rightarrow \infty} \mathbb{P}[|\mathcal{C}_n(x)| \geq \rho|\Lambda_n(x)| | x \in \mathcal{C}] = 1.$$

(ii) *For any  $\alpha' \in (\alpha, 2d)$  there exist  $\rho > 0$  and  $\ell_0$  such that for any  $\ell$  and  $n$  with  $\ell_0 \leq \ell \leq n/\ell_0$*

$$\mathbb{P}[|\mathcal{D}_n^{(\rho, \ell)}| \geq \rho|\Lambda_n(0)|] \geq 1 - e^{-\rho n^{2d-\alpha'}}.$$

This result can be interpreted as local clustering in the sense that with high probability (for large  $n$ ) particles are surrounded by many other particles belonging to the same connected network. Corollary 7 is the analog to Corollaries 3.3 and 3.4 in [8]. Once the proofs of Theorem 6 and Lemma 10 (a), below, are established it follows from the derivations in [8].

### 3.3 Graph distances

For  $x, y \in \mathbb{Z}^d$  we define  $d(x, y)$  to be the minimal number of occupied edges which connect  $x$  and  $y$ , and we set  $d(x, y) = \infty$  for  $y \notin \mathcal{C}(x)$ . The value  $d(x, y)$  is called graph distance or chemical distance between  $x$  and  $y$ , and it denotes the minimal number of occupied edges that need to be crossed from  $x$  to  $y$  (and vice versa). If  $d(x, y)$  is typically small for distant  $x$  and  $y$  then we say that the network has the small-world effect.

**Theorem 8** Assume  $\min\{\alpha, \beta\alpha\} > d$ .

(a) (infinite variance of degree distribution). Assume  $\tau = \beta\alpha/d < 2$ . For any  $\lambda > \lambda_c = 0$  there exists  $\eta_1 > 0$  such that for every  $\epsilon > 0$

$$\lim_{|x| \rightarrow \infty} \mathbb{P} \left[ \eta_1 \leq \frac{d(0, x)}{\log \log |x|} \leq (1 + \epsilon) \frac{2}{|\log(\beta\alpha/d - 1)|} \mid 0, x \in \mathcal{C} \right] = 1.$$

(b1) (finite variance of degree distribution case 1). Assume  $\tau = \beta\alpha/d > 2$  and  $\alpha \in (d, 2d)$ . For any  $\lambda > \lambda_c$  and any  $\epsilon > 0$

$$\lim_{|x| \rightarrow \infty} \mathbb{P} \left[ 1 - \epsilon \leq \frac{\log d(0, x)}{\log \log |x|} \leq (1 + \epsilon) \frac{\log 2}{\log(2d/\alpha)} \mid 0, x \in \mathcal{C} \right] = 1.$$

(b2) (finite variance of degree distribution case 2). Assume  $\min\{\alpha, \beta\alpha\} > 2d$ . There exists  $\eta_2 > 0$  such that

$$\lim_{|x| \rightarrow \infty} \mathbb{P} \left[ \eta_2 < \frac{d(0, x)}{|x|} \right] = 1.$$

From Theorem 8 (a) we conclude that in the case  $\tau \in (1, 2)$  we have a small-world effect and the graph distance is of order  $\log \log |x|$  as  $|x| \rightarrow \infty$ . In the case  $\tau > 2$  and  $\alpha \in (d, 2d)$  (for  $\lambda > \lambda_c$ ) the small-world effect is less pronounced in that the graph distance is conjectured to be of order  $(\log |x|)^\Delta$  for  $|x| \rightarrow \infty$ . Note that this is a conjecture because the bounds in Theorem 8 (b1) are not sufficiently sharp to obtain the exact constant  $\Delta > 0$ . Finally, in the case  $\min\{\alpha, \beta\alpha\} > 2d$  we do not have the small-world effect and graph distance behaves linearly in the Euclidean distance. In Figure 2 we illustrate Theorem 8 and we complete the conjectured picture about the graph distances.

Case (a) of Theorem 8 was proved in Theorems 5.1 and 5.3 of [13]. Statement (b1) proves upper and lower bounds in case 1 of finite variance of the degree distribution. The lower bound was proved in Theorem 5.5 of [13]. The upper bound will be proved below in Proposition 11. Finally, the lower bound in (b2) improves the one given in Theorem 5.6 of [13].

## 4 Discussion and outlook

In percolation theory one important problem is to understand the behavior of the model at criticality  $\lambda_c$ . In nearest-neighbor Bernoulli bond percolation on  $\mathbb{Z}^d$ , where nearest-neighbor

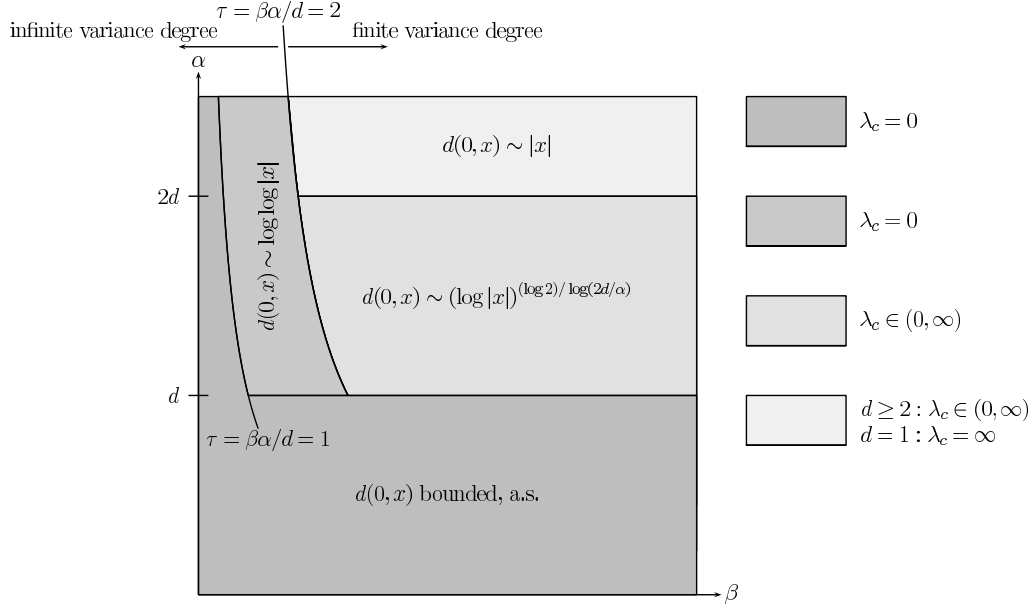


Figure 2: picture about the graph distances (partly as conjecture).

edges are vacant or occupied with probability  $p \in (0, 1)$ , it is known that for  $d = 2$  and for  $d \geq 19$  there is no percolation at criticality and hence the percolation function is continuous at the critical value (see [17] and [16] for more details). In cases  $3 \leq d \leq 18$  this question is still open. In the homogeneous long-range percolation model it was shown by [5] that there is no percolation at criticality for  $\alpha \in (d, 2d)$ . It is believed that the long-range percolation model behaves similarly to the nearest-neighbor Bernoulli percolation model when  $\alpha > 2d$  and, thus, showing continuity for such values and  $d > 1$  remains a difficult problem. In our model also the case  $\min\{\alpha, \beta\alpha\} > 2d$  for  $d > 1$  is open which is conjectured to behave as nearest-neighbor Bernoulli percolation, and hence is not of interest for real-life network modeling.

Another problem which remains to be answered in both homogeneous and inhomogeneous long-range models is the continuity of the critical parameter  $\lambda_c(\alpha)$  as a function of  $\alpha$  and also as a function of parameter  $\beta$ , the exponent of the power law in weights (in case of the inhomogeneous model). Moreover, for real-life network applications it will be important to (at least) get reasonable bounds on the critical value  $\lambda_c(\alpha)$  and the percolation probability  $\theta(\lambda, \alpha)$ . This will allow for model calibration of real-life networks so that (asymptotic) network properties can be studied.

There was quite some work done to understand the geometry of the homogeneous long-range percolation model. In particular, there are five different behaviors depending on  $\alpha < d, \alpha = d, \alpha \in (d, 2d), \alpha = 2d$  and  $\alpha > 2d$ , for a review of existing results see discussion in [10]. In some of the cases, like  $\alpha = 2d$  (for  $d \geq 1$ ) and  $\alpha > 2d$ , the results are not yet fully known. The case  $d = 1$  and  $\alpha = 2$  was resolved recently in [14]. It is clear that in the case of inhomogeneous long-range percolation the complexity even increases due to having more parameters and, hence,



degrees of freedom. For instance, the understanding of the graph distance behavior is still poor for  $\min\{\alpha, \beta\alpha\} > 2d$ , though we believe that it should behave similarly to nearest-neighbor Bernoulli bond percolation. Moreover, the optimal constants in the asymptotic behaviors of Theorem 8 are still open. However, we would like to emphasize that the inhomogeneous long-range percolation model fulfills the stylized fact that the degree distribution is heavy-tailed which is not the case for the homogeneous long-range percolation model. Therefore, the inhomogeneous model is an appealing framework for real-life network modeling, in particular for  $\tau \in (1, 2)$  where we obtain an infinite connected network for any  $\lambda > 0$ .

## 5 Proofs

### 5.1 Bounds on percolation on finite boxes

The basis for all the proofs of the previous statements is Lemma 9 below which determines large connected components on finite boxes.

**Lemma 9** *Assume  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$ . Choose  $\lambda \in (0, \infty)$  with  $\theta(\lambda, \alpha) > 0$  and let  $\alpha' \in [\alpha, 2d)$ . For every  $\varepsilon \in (0, 1)$  and  $\rho > 0$  there exists  $N_0 \geq 1$  such that for all  $m \geq N_0$*

$$\mathbb{P} \left[ |C_m| \geq \rho m^{\alpha'/2} \right] \geq 1 - \varepsilon,$$

where  $C_m$  is the largest connected component in box  $B_m = [0, m-1]^d$ .

**Sketch of proof of Lemma 9.** This lemma corresponds to Lemma 2.3 of [5] in our model. Its proof is based on renormalization arguments which only depend on the fact that  $\alpha \in (d, 2d)$  and that the edge probabilities are bounded from below by  $1 - \exp(-\lambda|x-y|^{-\alpha})$  for any  $x, y \in \mathbb{Z}^d$ . Using that  $W_x \geq 1$  for all  $x \in \mathbb{Z}^d$ , a.s., we see by stochastic dominance that the renormalization holds also true for our model. Renormalization shows that for  $m$  sufficiently large, the probability of  $\{B_m \text{ contains at least a positive fraction of } m^d \text{ vertices that are connected within a fixed enlargement of } B_m\}$  is bounded by a multiple of the probability of the same event but on a much smaller scale. To bound the latter probability we then use the fact that the model is percolating, and from this we can conclude Lemma 9. We skip the details of the proof of Lemma 9 and refer to the proof of Lemma 2.3 of [7] for the details, in particular, the bound on  $\psi_n$  in our homogeneous percolation model (see proof of Lemma 2.3 in [7]) also applies to the inhomogeneous percolation model.  $\square$

Although the above lemma does not allow the connected component  $C_m$  to have size proportional to the size of box  $B_m$ , it is useful because it allows to start a new renormalization scheme to improve these bounds. This results in our Theorem 6 and is done similar as in Section 3 of [8]. For the proof of Theorem 6 we use the following lemma which has two parts. The first one gives the initial step of the renormalization and the second one gives a standard site-bond percolation model result. Once the lemma is established the proof of Theorem 6 becomes a routine task.

Let  $C_m(x)$  denote the largest connected component in box  $B_m(x)$  (with a fixed deterministic rule if there is more than one largest connected component in  $B_m(x)$ ). For  $x, y \in m\mathbb{Z}^d$ , we say that boxes  $B_m(x)$  and  $B_m(y)$  are *pairwise attached*, write  $B_m(x) \Leftrightarrow B_m(y)$ , if there is an occupied edge between a vertex in  $C_m(x)$  and a vertex in  $C_m(y)$ .

**Lemma 10**

(a) Assume  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$ . Choose  $\lambda \in (0, \infty)$  such that  $\theta(\lambda, \alpha) > 0$ . For each  $\xi < \infty$  and  $r \in (0, 1)$  there exist  $m < \infty$  and an integer  $\delta > 0$  such that

$$\begin{aligned} \mathbb{P}[|C_m(x)| < \delta|B_m(x)|] &\leq 1 - r, \\ \mathbb{P}\left[B_m(x) \Leftrightarrow B_m(y) \mid |C_m(x)| \geq \delta|B_m(x)|, |C_m(y)| \geq \delta|B_m(y)|\right] &\geq 1 - e^{-\xi\left(\frac{|x-y|}{m}\right)^{-\alpha}}, \end{aligned}$$

for all  $x \neq y \in m\mathbb{Z}^d$ .

(b) [Lemma 3.6, [8]] Let  $d \geq 1$  and consider the site-bond percolation model on  $\mathbb{Z}^d$  with sites being alive with probability  $r \in [0, 1]$  and sites  $x, y \in \mathbb{Z}^d$  are attached with probability  $\tilde{p}_{x,y} = 1 - \exp(-\xi|x-y|^{-\alpha})$  where  $\alpha \in (d, 2d)$  and  $\xi \geq 0$ . Let  $|\tilde{\mathcal{C}}_N|$  be the size of the largest attached cluster  $\tilde{\mathcal{C}}_N$  of living sites in box  $B_N$ . For each  $\alpha' \in (\alpha, 2d)$  there exist  $N_0 \geq 1$ ,  $\nu > 0$  and  $\xi_0 < \infty$  such that

$$\mathbb{P}_{\xi,r}\left[|\tilde{\mathcal{C}}_N| \geq \nu|B_N|\right] \geq 1 - e^{-\nu\xi N^{2d-\alpha'}}$$

holds for all  $N \geq N_0$  whenever  $\xi \geq \xi_0$  and  $r \geq 1 - e^{-\nu\xi}$ .

**Proof of Lemma 10 (a).** We adapt the proof of Lemma 3.5 of [8] to our model. Fix  $r \in (0, 1)$  and  $\xi < \infty$ . Choose  $\rho > 0$  such that

$$\lambda(2\sqrt{d} + 1)^{-\alpha} \rho^2 = \xi,$$

note that this differs from choice (3.13) in [8]. Lemma 9 then provides that there exists  $N_0 \geq 1$  such that for all  $m \geq N_0$

$$\mathbb{P}[|C_m| < \rho m^{\alpha/2}] \leq 1 - r.$$

For the choice  $\delta = \rho m^{\alpha/2-d}$  the first part of the result follows. For the second part we choose  $x \neq y \in m\mathbb{Z}^d$ . For  $x' \in B_m(x)$  and  $y' \in B_m(y)$  we have upper bound, using that  $W_z \geq 1$  for all  $z \in \mathbb{Z}^d$ , a.s.,

$$1 - p_{x'y'} \leq \exp(-\lambda|x' - y'|^{-\alpha}) \leq \exp\left(-\lambda(2\sqrt{d} + 1)^{-\alpha}|x - y|^{-\alpha}\right), \quad (2)$$

a.s., where the latter no longer depends on the weights  $(W_z)_{z \in \mathbb{Z}^d}$ . For our choices of  $\delta$  and  $\rho$ , (2) implies

$$\begin{aligned} \mathbb{P}\left[B_m(x) \not\Leftrightarrow B_m(y) \mid |C_m(x)| \geq \delta|B_m(x)|, |C_m(y)| \geq \delta|B_m(y)|\right] \\ = \mathbb{E}\left[\prod_{x' \in C_m(x), y' \in C_m(y)} (1 - p_{x'y'}) \mid |C_m(x)| \geq \delta|B_m(x)|, |C_m(y)| \geq \delta|B_m(y)|\right] \\ \leq \exp\left(-\lambda(2\sqrt{d} + 1)^{-\alpha}|x - y|^{-\alpha} \rho^2 m^\alpha\right) = \exp\left(-\xi\left(\frac{|x - y|}{m}\right)^{-\alpha}\right). \end{aligned}$$

This shows the second inequality of part (a). For part (b) we refer to Lemma 3.6 in [8].  $\square$

**Proof of Theorem 6.** The proof follows as in Theorem 3.2 of [8], we briefly sketch the main argument. Choose the constants  $N_0 \geq 1$ ,  $\nu > 0$ ,  $\xi > \xi_0$ ,  $r \geq 1 - e^{-\nu\xi}$  and  $\delta > 0$  as in Lemma 10, and note that it is sufficient to prove the theorem for  $L = mN$ , where  $N \geq N_0$  and  $m$  is chosen (fixed) as in Lemma 10 (a). In this set up  $B_L$  can be viewed as a disjoint union of  $B_m(x)$  for  $x \in (m\mathbb{Z}^d \cap B_L)$ . There are  $N^d$  such disjoint boxes. We call

$B_m(x)$  alive if  $|C_m(x)| \geq \delta|B_m|$  and we say that disjoint  $B_m(x)$  and  $B_m(y)$  are pairwise attached if their largest connected components  $C_m(x)$  and  $C_m(y)$  share an occupied edge. Part (a) of Lemma 10 provides that  $B_m(x)$  is alive with probability exceeding  $r$  and  $B_m(x)$  and  $B_m(y)$  are pairwise attached with probability exceeding  $\tilde{p}_{x,y}$  for living boxes  $B_m(x)$  and  $B_m(y)$  with  $x, y \in m\mathbb{Z}^d$  (note that in this site-bond percolation model the attachedness property is only considered between living vertices because these form the clusters). For any  $N \geq N_0$ , let  $A_{N,m}$  be the event that box  $B_L$  contains a connected component formed by attaching at least  $\nu|B_N|$  of the living boxes. On event  $A_{N,m}$  we have for the largest connected component in  $B_L$

$$|C_L| \geq (\nu|B_N|)(\delta|B_m|) = \nu\delta|B_L|,$$

thus, the volume of the largest connected component  $C_L$  in box  $B_L$  is proportional to the volume of that box and there remains to show that this occurs with sufficiently large probability. Part (b) of Lemma 10 and stochastic dominance provides (note that we scale  $x, y \in m\mathbb{Z}^d$  from Lemma 10 (a) to the site-bond percolation model on  $\mathbb{Z}^d$  in Lemma 10 (b))

$$\begin{aligned} \mathbb{P}[|C_L| \geq \nu\delta|B_L|] &\geq \mathbb{P}[A_{N,m}] \geq \mathbb{P}_{\xi,r} \left[ |\tilde{C}_N| \geq \nu|B_N| \right] \\ &\geq 1 - e^{-\nu\xi N^{2d-\alpha'}} = 1 - e^{-\nu\xi m^{\alpha'} - 2dL^{2d-\alpha'}}. \end{aligned}$$

Choosing  $\rho \leq \min\{\nu\delta, \nu\xi m^{\alpha'-2d}\}$  provides

$$\mathbb{P}[|C_L| \geq \rho|B_L|] \geq 1 - e^{-\rho L^{2d-\alpha'}}.$$

This finishes the proof of Theorem 6.  $\square$

**Proof of Corollary 7.** The proofs of (i) and (ii) of Corollary 7 follow completely analogous to the proofs of Corollaries 3.3 and 3.4 in [8] (note that Lemma 10 (a) replaces Lemma 3.5 of [8] and Theorem 6 replaces Theorem 3.2 of [8]).  $\square$

## 5.2 Proof of continuity of the percolation probability

The key to the proofs of the continuity statements is again Lemma 9.

**Proof of Theorem 3.** Note that  $\min\{\alpha, \beta\alpha\} > d$  and  $\alpha \in (d, 2d)$  imply that  $\lambda_c < \infty$ . Therefore, there exists  $\lambda \in (\lambda_c, \infty)$  with  $\theta = \theta(\lambda, \alpha) > 0$ . For these choices of  $\lambda > 0$  we have a unique infinite cluster  $\mathcal{C}$ , a.s., and we can apply Lemma 9.

We consider the same site-bond percolation model on  $\mathbb{Z}^d$  as in Lemma 10 (b). Choose  $\alpha' \in (\alpha, 2d)$ ,  $0 < \chi < 1 - \varepsilon < 1$  and  $\kappa > 0$  and define the model as follows: the following events are independent and every site  $x \in \mathbb{Z}^d$  is alive with probability  $r = 1 - \varepsilon - \chi \in (0, 1)$  and sites  $x, y \in \mathbb{Z}^d$  are attached with probability  $\tilde{p}_{xy} = 1 - \exp(-\kappa(1 - \chi)|x - y|^{-\alpha'})$ . For given  $\alpha' \in (\alpha, 2d)$  we choose the parameters  $\varepsilon, \chi, \kappa$  such that there exists an infinite attached cluster of living vertices, a.s., which is possible (see proof of Theorem 2.5 in [5]).

The proof is now similar to the one of Theorem 6. Choose  $\rho > 0$  such that  $\lambda(2\sqrt{d} + 1)^{-\alpha'}\rho^2 = \kappa$ . From Lemma 9 we know that for all  $m$  sufficiently large and any  $x \in m\mathbb{Z}^d$

$$\mathbb{P}[|C_m(x)| \geq \rho m^{\alpha'/2}] \geq 1 - \varepsilon > 1 - \varepsilon - \chi = r,$$

where  $C_m(x)$  denotes the largest connected component in  $B_m(x)$ . The latter events define alive vertices  $x$  on the lattice  $m\mathbb{Z}^d$  (which due to scaling is equivalent to the above aliveness in the site-bond percolation model on  $\mathbb{Z}^d$ ). Note that this aliveness property is independent between different vertices  $x \in m\mathbb{Z}^d$ . Attachedness  $B_m(x) \Leftrightarrow B_m(y)$ , for  $x \neq y \in m\mathbb{Z}^d$ , is then used as in the proof of Theorem 6 and we obtain in complete analogy

to the proof of the latter theorem

$$\begin{aligned} & \mathbb{P} \left[ B_m(x) \Leftrightarrow B_m(y) \mid |C_m(x)| \geq \rho m^{\alpha'/2}, |C_m(y)| \geq \rho m^{\alpha'/2} \right] \\ & \geq 1 - \exp \left( -\lambda \left( 2\sqrt{d} + 1 \right)^{-\alpha} |x - y|^{-\alpha} \rho^2 m^{\alpha'} \right) \geq 1 - \exp \left( -\kappa \left( \frac{|x - y|}{m} \right)^{-\alpha'} \right), \end{aligned}$$

where in the last step we used the choice of  $\rho$  and the fact that  $\alpha < \alpha'$ . Since  $\kappa > \kappa(1 - \chi)$  we get percolation and there exists an infinite cluster  $\mathcal{C}$ , a.s., which implies  $\theta(\lambda, \alpha) > 0$ . Of course, this is no surprise because of the choice  $\lambda > \lambda_c$  with  $\theta(\lambda, \alpha) > 0$ .

Note that the probability of a vertex  $x \in m\mathbb{Z}^d$  being alive depends only on finitely many edges of maximal distance  $\sqrt{d}m$  (they all lie in the box  $B_m(x)$ ) and therefore this probability is a continuous function of  $\lambda$  and  $\alpha$ . This implies that we can choose  $\delta \in (0, \chi\lambda)$  and  $\gamma \in (0, \alpha' - \alpha)$  so small that

$$\mathbb{P}_{\lambda-\delta, \alpha+\gamma} \left[ |C_m(x)| \geq \rho m^{\alpha'/2} \right] \geq 1 - \varepsilon - \chi = r,$$

where  $\mathbb{P}_{\lambda-\delta, \alpha+\gamma}$  is the measure where for occupied edges we replace parameters  $\lambda$  by  $\lambda - \delta \in (0, \lambda)$  and  $\alpha$  by  $\alpha + \gamma \in (\alpha, \alpha')$ . As in (3) we obtain, note  $\alpha + \gamma < \alpha'$ ,

$$\begin{aligned} & \mathbb{P}_{\lambda-\delta, \alpha+\gamma} \left[ B_m(x) \Leftrightarrow B_m(y) \mid |C_m(x)| \geq \rho m^{\alpha'/2}, |C_m(y)| \geq \rho m^{\alpha'/2} \right] \\ & \geq 1 - \exp \left( -(\lambda - \delta) \left( 2\sqrt{d} + 1 \right)^{-(\alpha+\gamma)} |x - y|^{-(\alpha+\gamma)} \rho^2 m^{\alpha'} \right) \\ & \geq 1 - \exp \left( -\kappa (1 - \delta/\lambda) \left( \frac{|x - y|}{m} \right)^{-\alpha'} \right). \end{aligned}$$

Since  $\delta/\lambda < \chi$  we get percolation and there exists an infinite cluster  $\mathcal{C}$ , a.s., which implies that  $\theta(\lambda - \delta, \alpha + \gamma) > 0$ . This finishes the proof of Theorem 3.  $\square$

**Proof of Theorem 5.** We need to modify Proposition 1.3 of [1] because in our model edges are not occupied independently induced by the random choices of weights  $(W_x)_{x \in \mathbb{Z}^d}$ .

- (i) From Theorem 3 it follows that  $\theta(\lambda, \alpha) = 0$  for all  $\lambda \in (0, \lambda_c]$ , which proves continuity of  $\lambda \mapsto \theta(\lambda, \alpha)$  on  $(0, \lambda_c]$ .
- (ii) Next we show that  $\lambda \mapsto \theta(\lambda, \alpha)$  is left-continuous on  $\lambda > \lambda_c$ , that is,

$$\lim_{\lambda' \uparrow \lambda} \theta(\lambda', \alpha) = \theta(\lambda, \alpha). \quad (3)$$

To prove this we couple all percolation realization as  $\lambda$  varies. This is achieved by randomizing the percolation constant  $\lambda$ , see [1] and [22]. Conditionally given the i.i.d. weights  $(W_x)_{x \in \mathbb{Z}^d}$ , define a collection of independent exponentially distributed random variables  $\phi_{(x,y)}$ , indexed by the edges  $(x, y)$ , which have conditional distribution

$$\mathbf{P} \left[ \phi_{(x,y)} \leq \ell \mid (W_x)_{x \in \mathbb{Z}^d} \right] = 1 - \exp \left( -\frac{\ell W_x W_y}{|x - y|^\alpha} \right), \quad \ell \in (0, \infty), \quad (4)$$

compare to (1). We denote the probability measure of  $(\phi_{(x,y)})_{x,y \in \mathbb{Z}^d}$  by  $\mathbf{P}$  in order to distinguish this coupling model. We say that an edge  $(x, y)$  is  $\ell$ -open if  $\phi_{(x,y)} < \ell$ , and we define the connected cluster  $C_\ell(0)$  of the origin to be the set of all vertices  $x \in \mathbb{Z}^d$  which are connected to the origin by an  $\ell$ -open path. Note that we have a natural ordering in  $\ell$ , i.e. for  $\ell_1 < \ell_2$  we obtain  $C_{\ell_1}(0) \subset C_{\ell_2}(0)$ . Moreover for  $\ell = \lambda > 0$ , the  $\lambda$ -open edges are exactly the occupied edges in this coupling (note that the exponential distribution (4) is absolutely continuous). This implies for  $\ell = \lambda$

$$\theta(\lambda, \alpha) = \mathbb{P}[|\mathcal{C}(0)| = \infty] = \mathbf{P}[|C_\lambda(0)| = \infty].$$

By countable subadditivity of  $\mathbf{P}$  and the increasing property of  $C_\ell(0)$  in  $\ell$  we have

$$\lim_{\lambda' \uparrow \lambda} \theta(\lambda', \alpha) = \mathbf{P}[|C_{\lambda'}(0)| = \infty \text{ for some } \lambda' < \lambda].$$

Moreover, the increasing property of  $C_\ell(0)$  in  $\ell$  provides  $\{|C_{\lambda'}(0)| = \infty \text{ for some } \lambda' < \lambda\} \subset \{|C_\lambda(0)| = \infty\}$ . Therefore, to prove (3) it suffices to show that

$$\mathbf{P} \left[ \{|C_{\lambda'}(0)| < \infty \text{ for all } \lambda' < \lambda\} \cap \{|C_\lambda(0)| = \infty\} \right] = 0.$$

Choose  $\lambda_0 \in (\lambda_c, \lambda)$ . Since there is a unique infinite cluster for  $\lambda_0 > \lambda_c$ , a.s., there exists an infinite cluster  $C_{\lambda_0} \subset C_\lambda(0)$  on the set  $\{|C_\lambda(0)| = \infty\}$ . If the origin belongs to  $C_{\lambda_0}$  then the proof is done. Otherwise, because  $C_{\lambda_0}$  is a subgraph of  $C_\lambda(0)$ , there exists a finite path  $\pi$  of  $\lambda$ -open edges connecting the origin with an edge in  $C_{\lambda_0}$ . By the definition of  $\lambda$ -open edges we have  $\phi_{(x,y)} < \lambda$  for all edges  $(x,y) \in \pi$ . Since  $\pi$  is finite we obtain the strict inequality  $\lambda_1 = \max_{(x,y) \in \pi} \phi_{(x,y)} < \lambda$ . Choose  $\lambda' \in (\lambda_0 \vee \lambda_1, \lambda)$  and it follows that  $|C_{\lambda'}(0)| = \infty$ . This completes the proof for the left-continuity in  $\lambda$ .

(iii) Finally, we need to prove right-continuity of  $\lambda \mapsto \theta(\lambda, \alpha)$  on  $\lambda \geq \lambda_c$ . For integers  $n > 1$  we consider the boxes  $\Lambda_n = [-n, n]^d$  centered at the origin, see also Corollary 7. We define the events  $A_n = \{\mathcal{C}(0) \cap \Lambda_n^c \neq \emptyset\}$ , i.e., the connected component  $\mathcal{C}(0)$  of the origin leaves the box  $\Lambda_n = [-n, n]^d$ . Note that  $\theta(\lambda, \alpha)$  is the decreasing limit of  $\mathbb{P}[A_n]$  as  $n \rightarrow \infty$ . Therefore, it suffices to show that  $\mathbb{P}[A_n]$  is a continuous function in  $\lambda$ . We write  $\mathbb{P}_\lambda = \mathbb{P}$  to indicate on which parameter  $\lambda$  the probability law depends. We again denote by  $\mathcal{C}_n(0)$  the connected component of the origin connected within box  $\Lambda_n$ , see Corollary 7. Then, we have

$$A_n = \{\mathcal{C}(0) \cap \Lambda_n^c \neq \emptyset\} = \{\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c\}.$$

Choose  $\delta_0 \in (0, \lambda)$ , then we have for all  $\lambda' \in (\lambda - \delta_0, \lambda + \delta_0)$  and all  $n' > n$

$$\begin{aligned} |\mathbb{P}_\lambda[A_n] - \mathbb{P}_{\lambda'}[A_n]| &= |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c]| \\ &\leq |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})]| \\ &\quad + 2 \mathbb{P}_{\lambda+\delta_0}[\mathcal{C}_n(0) \Leftrightarrow \Lambda_{n'}^c] \\ &\leq |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})]| \\ &\quad + 2(2n+1)^d \sup_{x \in \Lambda_n} \mathbb{P}_{\lambda+\delta_0}[x \Leftrightarrow \Lambda_{n'}^c]. \end{aligned} \tag{5}$$

We bound the two terms on the right-hand side of (5).

(a) First we prove that for all  $\varepsilon > 0$  there exists  $n' > n$  such that for all  $x \in \Lambda_n$

$$\mathbb{P}_{\lambda+\delta_0}[x \Leftrightarrow \Lambda_{n'}^c] < \varepsilon(2n+1)^{-d}/4. \tag{6}$$

This is done as follows. For  $m > n$  we define the following events

$$L_m = \{x \Leftrightarrow \partial\Lambda_{m+1}\} = \{x \Leftrightarrow (\Lambda_{m+1} \setminus \Lambda_m)\}.$$

This implies for  $n' > n$  that

$$E_{n'} \stackrel{\text{def.}}{=} \{x \Leftrightarrow \Lambda_{n'}^c\} = \bigcup_{m \geq n'} L_m.$$

Moreover, note that  $E_{n'}$  is decreasing in  $n'$ ,

$$\limsup_{n' \rightarrow \infty} \mathbb{P}_{\lambda+\delta_0}[E_{n'}] = \lim_{n' \rightarrow \infty} \mathbb{P}_{\lambda+\delta_0}[E_{n'}] = \mathbb{P}_{\lambda+\delta_0} \left[ \bigcap_{n' > n} E_{n'} \right] = \mathbb{P}_{\lambda+\delta_0} \left[ \bigcap_{n' > n} \left( \bigcup_{m \geq n'} L_m \right) \right].$$

We prove (6) by contradiction. Assume that (6) does not hold true, i.e.  $\limsup_{n' \rightarrow \infty} \mathbb{P}[E_{n'}] > 0$ . Then the first lemma of Borel-Cantelli implies

$$\infty = \sum_{m > n} \mathbb{P}_{\lambda+\delta_0}[L_m] = \sum_{m > n} \mathbb{P}_{\lambda+\delta_0}[x \Leftrightarrow (\Lambda_{m+1} \setminus \Lambda_m)] = \mathbb{E}_{\lambda+\delta_0} \left[ \sum_{m > n} 1_{\{x \Leftrightarrow (\Lambda_{m+1} \setminus \Lambda_m)\}} \right].$$

The latter implies that the degree distribution  $D_x = |\{y \in \mathbb{Z}^d; x \Leftrightarrow y\}|$  has an infinite mean. This is a contradiction to Theorem 2.2 of [13] saying that for  $\min\{\alpha, \beta\alpha\} > d$  the survival function of the degree distribution has a power-law decay with rate  $\alpha\beta/d > 1$  which provides a finite mean. Therefore, claim (6) holds true.

(b) For all  $\varepsilon > 0$  and all  $n' > n$  there exists  $\delta_1 \in (0, \delta_0)$  such that for all  $\lambda' \in (\lambda - \delta_1, \lambda + \delta_1)$

$$|\mathbb{P}_\lambda [\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})] - \mathbb{P}_{\lambda'} [\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})]| < \varepsilon/2. \quad (7)$$

Note that  $\Lambda_{n'}$  only contains finitely many edges of finite distance. Therefore, continuity in  $\lambda$  is straightforward which provides claim (7).

Combining (6) and (7) provides continuity of  $\mathbb{P}_\lambda[A_n]$  in  $\lambda$  for all  $n$ , see also (5). Therefore, right-continuity of  $\lambda \mapsto \theta(\lambda, \alpha)$  follows. This finishes the proof of Theorem 5.  $\square$

### 5.3 Proofs of the graph distances

In this section we prove Theorem 8. Statement (a) of Theorem 8 is proved in Theorems 5.1 and 5.3 of [13], the lower bound of statement (b1) is proved in Theorem 5.5 of [13]. Therefore, there remain the proofs of the upper bound in (b1) and of the lower bound in (b2) of Theorem 8.

The proof of the upper bound in Theorem 8 (b1) follows from the following proposition and the fact that  $\alpha \mapsto \Delta(\alpha, 2d) = \log 2 / \log(2d/\alpha)$  is a continuous function. The following proposition corresponds to Proposition 4.1 in [8] in the homogeneous long-range percolation model.

**Proposition 11** *Let  $\alpha \in (d, 2d)$  and  $\tau = \beta\alpha/d > 2$  and  $\lambda > \lambda_c$ . For each  $\Delta' > \Delta = \Delta(\alpha, 2d) = \log 2 / \log(2d/\alpha)$  and each  $\varepsilon > 0$ , there exists  $N_0 < \infty$  such that*

$$\mathbb{P} \left[ d(x, y) \geq (\log |x - y|)^{\Delta'}, x, y \in \mathcal{C} \right] \leq \varepsilon$$

*holds for all  $x, y \in \mathbb{Z}^d$  with  $|x - y| \geq N_0$ .*

**Sketch of proof of Proposition 11.** We only sketch the proof because it is almost identical to the one in [8]. Definition 1 and Figure 1 of [8] defines for  $x, y \in \mathbb{Z}^d$  a hierarchy of depth  $m \in \mathbb{N}$  connecting  $x$  and  $y$  as the following collection of vertices:

$$\mathcal{H}_m(x, y) = \left\{ z_\sigma \in \mathbb{Z}^d; \sigma \in \{0, 1\}^k \text{ for } k = 1, \dots, m \right\},$$

is a hierarchy of depth  $m \in \mathbb{N}$  connecting  $x$  and  $y$  if

- (1)  $z_0 = x$  and  $z_1 = y$ ,
- (2)  $z_{\sigma 00} = z_{\sigma 0}$  and  $z_{\sigma 11} = z_{\sigma 1}$  for all  $k = 0, \dots, m - 2$  and  $\sigma \in \{0, 1\}^k$ ,
- (3) for all  $k = 0, \dots, m - 2$  and  $\sigma \in \{0, 1\}^k$  such that  $z_{\sigma 01} \neq z_{\sigma 10}$  the edge between  $z_{\sigma 01}$  and  $z_{\sigma 10}$  is occupied,
- (4) each bond  $(z_{\sigma 01}, z_{\sigma 10})$  specified in (3) appears only once in  $\mathcal{H}_k(x, y)$ .

The pairs of vertices  $(z_{\sigma 00}, z_{\sigma 01})$  and  $(z_{\sigma 10}, z_{\sigma 11})$  are called gaps. The proof is then based on the fact that for large distances  $|x - y|$  the event  $\mathcal{B}_m$  of the existence of a hierarchy  $\mathcal{H}_m(x, y)$  of depth  $m$  that connects  $x$  and  $y$  through points  $z_\sigma$  which are dense is very likely ( $m$  appropriately chosen), see Lemma 4.3 in [8], in particular formula (4.18) in [8] (where the key is Corollary 7 (ii)). On this likely event  $\mathcal{B}_m$ , Lemma 4.2 of [8] then proves that the graph distance cannot be too large, see (4.8) in [8]. We can now almost literally translate Lemmas 4.2 and 4.3 of [8] to our situation. The only changes are that in formulas (4.16) and (4.21) of [8] we need to replace  $\beta > 0$  of [8]'s notation by  $\lambda$  in our notation and we need to use that the weights  $W_x$  are at least one, a.s. We refrain from giving more details.  $\square$

There remains the proof of the lower bound in (b2) of Theorem 8. We use a renormalization technique which is based on a scheme introduced by [6]. Choose an integer valued sequence  $a_n > 1$ ,  $n \in \mathbb{N}_0$ , and define the box lengths  $(m_n)_{n \in \mathbb{N}_0}$  as follows: set  $m_0 = a_0$  and for  $n \in \mathbb{N}$ ,

$$m_n = a_n m_{n-1} = m_0 \prod_{i=1}^n a_i = \prod_{i=0}^n a_i.$$

Define the  $n$ -stage boxes,  $n \in \mathbb{N}_0$ , by

$$B_{m_n}(x) = x + [0, m_n - 1]^d, \quad \text{for } x \in \mathbb{Z}^d.$$

For  $n \geq 1$ , the children of  $n$ -stage box  $B_{m_n}(x)$  are the  $a_n^d$  disjoint  $(n-1)$ -stage boxes

$$B_{m_{n-1}}(x + y m_{n-1}) = x + y m_{n-1} + [0, m_{n-1} - 1]^d \subset \mathbb{Z}^d \quad \text{with } y \in ([0, a_n - 1]^d \cap \mathbb{Z}^d).$$

We are going to define *good*  $n$ -stage boxes  $B_{m_n}(\cdot)$ , note that we need a different definition from Definition 2 of [6].

**Definition 12 (good  $n$ -stage boxes)** Choose  $n \in \mathbb{N}_0$  and  $x \in \mathbb{Z}^d$  fixed.

- 0-stage box  $B_{m_0}(x)$  is good under a given edge configuration if there is no occupied edge in  $B_{m_0}(x)$  with size larger than  $m_0/100$ .
- $n$ -stage box  $B_{m_n}(x)$ ,  $n \geq 1$ , is good under a given edge configuration if for all  $j \in \{-1, 0, 1\}^d$ 
  - (a) there is no occupied edge in  $B_{m_n}(x + j \frac{m_{n-1}}{2})$  with size larger than  $m_{n-1}/100$ ; and
  - (b) among the children of  $B_{m_n}(x + j \frac{m_{n-1}}{2})$  there are at most  $3^d$  that are not good.

**Lemma 13** Assume  $\min\{\alpha, \beta\alpha\} > d$ . For all  $\delta \in (0, \alpha(\beta \wedge 1) - d)$  there exist  $t_0 \geq 1$  and a constant  $c_1 > 0$  such that for all  $t \geq t_0$  and all  $s \geq 1$ ,

$$\mathbb{P} \left[ \text{there is an occupied edge in } [0, s-1]^d \text{ with size larger than } t \right] \leq c_1 s^d t^{d-\alpha(\beta \wedge 1)+\delta}.$$

**Proof of Lemma 13.** Let  $W_1$  and  $W_2$  be two independent random variables each having a Pareto distribution with parameters  $\theta = 1$  and  $\beta > 0$ . For  $u \geq 1$  we have, using integration by parts in the first step,

$$\begin{aligned} \mathbb{E} \left[ \frac{W_1 W_2}{u} \wedge 1 \right] &= \frac{1}{u} + \frac{1}{u} \int_1^u \mathbb{P}[W_1 W_2 > v] dv = \frac{1}{u} + \frac{1}{u} \int_1^u v^{-\beta} (1 + \beta \log v) dv \\ &\leq (1 + \beta \log u) \left( u^{-(\beta \wedge 1)} + \frac{1}{u} \int_1^u v^{-\beta} dv \right) \\ &\leq \max\{1 + \log u, 1 + 1_{\{\beta \neq 1\}}/|\beta - 1|\} (1 + \beta \log u) u^{-(\beta \wedge 1)}, \end{aligned}$$

where the last step follows by distinguishing between the cases  $\beta = 1$ ,  $\beta > 1$  and  $\beta < 1$ . Choose  $t_0$  so large that  $\lambda^{-1} t_0^\alpha \geq 1$  which, together with the above calculations, implies that for all  $t \geq t_0$  and  $x, y \in \mathbb{Z}^d$  with  $|x - y| > t \geq t_0$ ,

$$\begin{aligned} \mathbb{E} \left[ \frac{\lambda W_x W_y}{|x - y|^\alpha} \wedge 1 \right] &\leq (1 + 1_{\{\beta \neq 1\}}/|\beta - 1|) (1 + \max\{1, \beta\} \log(\lambda^{-1} |x - y|^\alpha))^2 (\lambda^{-1} |x - y|^\alpha)^{-(\beta \wedge 1)} \\ &\leq |x - y|^{-\alpha(\beta \wedge 1)+\delta}, \end{aligned}$$

where the second inequality holds for all  $|x - y| > t \geq t_0$  with  $t_0$  large enough. It follows that for all  $t \geq t_0$ , using  $1 - e^{-x} \leq x \wedge 1$ ,

$$\begin{aligned} \mathbb{P} \left[ \text{there is an occupied edge in } [0, s-1]^d \text{ with size larger than } t \right] &\leq \sum_{\substack{x, y \in [0, s-1]^d: \\ |x-y| > t}} \mathbb{E} \left[ \frac{\lambda W_x W_y}{|x-y|^\alpha} \wedge 1 \right] \\ &\leq \sum_{\substack{x, y \in [0, s-1]^d: \\ |x-y| > t}} |x-y|^{-\alpha(\beta \wedge 1) + \delta} \leq s^d \sum_{y \in \mathbb{Z}^d: |y| > t} |y|^{-\alpha(\beta \wedge 1) + \delta}. \end{aligned}$$

Hence, for an appropriate constant  $c_1 > 0$  and for all  $t \geq t_0$  with  $t_0$  sufficiently large,

$$\mathbb{P} \left[ \text{there is an occupied edge in } [0, s-1]^d \text{ with size larger than } t \right] \leq c_1 s^d t^{d - \alpha(\beta \wedge 1) + \delta},$$

which finishes the proof of Lemma 13.  $\square$

**Lemma 14** Assume  $\min\{\alpha, \beta\alpha\} > 2d$ . For  $a_n = n^2$ ,  $n \geq 1$ , and  $a_0$  sufficiently large we have

$$\sum_{n \geq 0} \mathbb{P} [B_{m_n}(0) \text{ is not good}] < \infty.$$

This lemma is the analog in our model to Lemma 1 of [6] and provides a Borel-Cantelli type of result that eventually the boxes  $B_{m_n}(0)$  are good, a.s., for all  $n$  sufficiently large.

**Proof of Lemma 14.** We prove by induction that  $\psi_n = \mathbb{P} [B_{m_n}(0) \text{ is not good}]$  is summable. Choose  $\delta \in (0, \alpha(\beta \wedge 1) - 2d)$  and set  $\gamma = \min\{\alpha, \beta\alpha\} - 2d - \delta > 0$ . For  $m_0$  sufficiently large we obtain by Lemma 13,

$$\begin{aligned} \psi_0 &= \mathbb{P} [\text{there is an occupied edge in } B_{m_0}(0) \text{ with size larger than } m_0/100] \\ &\leq c_1 m_0^d \left( \frac{m_0}{100} \right)^{d - \alpha(\beta \wedge 1) + \delta} < 3^{-d} 2^{-4d-1} e^{-2}, \end{aligned} \quad (8)$$

where the last step holds true for  $m_0$  sufficiently large. Because  $B_{m_1}(0)$  has only one child (because  $a_1 = 1$ ) we get for  $m_0$  sufficiently large

$$\psi_1 \leq 3^d \psi_0 \leq c_1 3^d m_0^d \left( \frac{m_0}{100} \right)^{d - \alpha(\beta \wedge 1) + \delta} < 3^{-d} 2^{-8d-1} e^{-4}. \quad (9)$$

For the induction step we note that  $n$ -stage box  $B_{m_n}(0)$  is not good if at least one of the  $3^d$  translations  $B_{m_n}(0 + j \frac{m_{n-1}}{2})$ ,  $j \in \{-1, 0, 1\}^d$ , fails to have property (a) or (b) of Definition 12. Using translation invariance and Lemma 13 we get for all  $n \geq 2$  and for all  $m_0$  sufficiently large, set  $c_2 = c_1 100^{\alpha(\beta \wedge 1) - d - \delta}$ ,

$$\psi_n \leq 3^d \left( c_2 a_n^{\alpha(\beta \wedge 1) - d - \delta} m_n^{-\gamma} + \mathbb{P} [\text{there are at least } 3^d + 1 \text{ children of } B_{m_n}(0) \text{ that are not good}] \right).$$

Note that the event in the probability above ensures that there are at least two children  $B_{m_{n-1}}(y)$  and  $B_{m_{n-1}}(z)$  of  $B_{m_n}(0)$  that are not good and are separated by at least Euclidean distance  $2m_{n-1}$ . Therefore, using  $m_i = a_0(i!)^2$ ,  $i \geq 0$ , the two boxes  $B_{m_{n-1}}(y)$  and  $B_{m_{n-1}}(z)$  are well separated in the sense that the events  $\{B_{m_{n-1}}(y) \text{ is not good}\}$  and  $\{B_{m_{n-1}}(z) \text{ is not good}\}$  are independent. Note that for the latter we need to make sure that  $B_{m_{n-1}}(y + jm_{n-2}/2)$  and  $B_{m_{n-1}}(z + lm_{n-2}/2)$  are disjoint for all  $j, l \in \{-1, 0, 1\}^d$ , which is the case because  $B_{m_{n-1}}(y)$  and  $B_{m_{n-1}}(z)$  have at least distance  $2m_{n-1}$ . The independence implies the following bound

$$\begin{aligned} \psi_n &\leq 3^d \left( c_2 a_n^{\alpha(\beta \wedge 1) - d - \delta} m_n^{-\gamma} + \binom{a_n^d}{2} \psi_{n-1}^2 \right) \leq 3^d \left( c_2 a_n^{\alpha(\beta \wedge 1) - d - \delta} m_n^{-\gamma} + a_n^{2d} \psi_{n-1}^2 \right) \\ &= 3^d \left( c_2 n^{2(\gamma+d)} (m_0(n!)^2)^{-\gamma} + n^{4d} \psi_{n-1}^2 \right) = c_2 3^d m_0^{-\gamma} n^{2(\gamma+d)} (n!)^{-2\gamma} + 3^d n^{4d} \psi_{n-1}^2. \end{aligned}$$

It follows that there is  $n_0 < \infty$  such that for all  $n \geq n_0$  and  $m_0$  large enough

$$\psi_n \leq 3^{-d} 2^{-4d-2} e^{-2} (n+1)^{-4d} e^{-2n} + 3^d n^{4d} \psi_{n-1}^2, \quad (10)$$



and we can choose  $m_0$  so large that (10) holds true also for all  $2 \leq n < n_0$ . We claim that for all  $a_0 = m_0$  sufficiently large and all  $n \geq 0$ ,

$$\psi_n \leq 3^{-d} 2^{-4d-1} e^{-2} (n+1)^{-4d} e^{-2n}, \quad (11)$$

which will imply Lemma 14 because the right-hand side is summable. Indeed, (11) is true for  $n \in \{0, 1\}$  by (8) and (9). Assuming that (11) holds for all  $n-1$  with  $n \geq 2$  we get, using (10),

$$\begin{aligned} \psi_n &\leq 3^{-d} 2^{-4d-2} e^{-2} (n+1)^{-4d} e^{-2n} + 3^d n^{4d} \psi_{n-1}^2 \\ &\leq 3^{-d} 2^{-4d-2} e^{-2} (n+1)^{-4d} e^{-2n} + 3^{-d} n^{-4d} 2^{-8d-2} e^{-4} e^{-4n+4} \\ &= 3^{-d} 2^{-4d-1} e^{-2} (n+1)^{-4d} e^{-2n} \left( 2^{-1} + \left( \frac{n+1}{n} \right)^{4d} 2^{-4d-1} e^{-2n+2} \right) \\ &\leq 3^{-d} 2^{-4d-1} e^{-2} (n+1)^{-4d} e^{-2n} (2^{-1} + 2^{-1}), \end{aligned}$$

where the last step follows since  $(n+1)/n \leq 2$  and  $e^{-2n+2} \leq 1$ .  $\square$

The following lemma is the analog of Proposition 3 of [6] and it depends on Lemma 2 of [6] and Lemma 14. Since its proof is completely similar to the one of Proposition 3 of [6] once Lemma 14 has been established we skip this proof.

**Lemma 15 (Proposition 3 of [6])** *Choose  $a_n = n^2$  for  $n \geq 1$ . There exists a constant  $c_3 > 0$  such that for every  $n$  sufficiently large, if for every  $j \in \{-1, 0, 1\}^d$  the  $n$ -stage box  $B_{m_n}(0 + j \frac{m_n}{2})$  is good and for every  $l > n$  the  $l$ -stage boxes  $\hat{B}_{m_l}$  centered at  $B_{m_n}(0)$  are good, then if  $x, y \in B_{m_n}(0)$  satisfy  $|x - y| > m_n/8$  then  $d(x, y) \geq c_3|x - y|$ .*

**Proof of Theorem 8 (b2).** Lemma 14 says that, a.s., the  $l$ -stage boxes  $\hat{B}_{m_l}$  are eventually good for all  $l \geq n$ . Moreover, from Lemma 15 we obtain the linearity in the distance for these good boxes which says that, a.s., for  $n$  sufficiently large and  $|x| > m_n/8$  we have  $d(0, x) \geq c_3|x|$ .  $\square$

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